Student's Name: ________________________ Teacher's Code: ________________

Saint Ignatius' College, Riverview

Mathematics Assessment Task

2024

Year 12
Mathematics (Extension One)
Task 4
Trial HSC Examination
Date: 21st August 2024

General Instructions:

- **Reading time: 10 minutes**
- **Time Allowed: 2 hours.**
- Write using a black pen.
- Calculators approved by NESA may be used.
- Attempt all questions in the booklets provided.
- Write **your name** and **your teacher's code** in the positions indicated.
- Marks may not be awarded for missing or carelessly arranged work.

Teachers:

- Mr R Maxwell **REM**
- Mr D Reidy **DPR**
- Mr N Mushan **NHM**
- Mr J Newey **JPN**

Topics Examined:

Section A	**10 Marks**
Multiple Choice	
Section B	
Short Answer	
Question 11	**15 Marks**
Question 12	**15 Marks**
Question 13	**15 Marks**
Question 14	**15 Marks**
Total	**70 Marks**

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SECTION I: Multiple Choice Questions:

1. How many integer solutions are there for the inequality $\dfrac{1}{|x-2|} > \dfrac{1}{4}$?

A. 7

B. 6

C. 5

D. 4

2. Consider the equation $3x^3 + 2x^2 - x + 5 = 0$.
What is the sum of the reciprocals of the roots of this equation?

A. $\dfrac{1}{5}$

B. $\dfrac{2}{5}$

C. $\dfrac{5}{2}$

D. -5

3. What is the maximum value of $15sin\theta - 8cos\theta + 2$?

A. 9

B. 17

C. 19

D. 23

4. Find the cartisian equation of the curve represented by the parametric equations:

$x = 4 - t$ and $y = 3t^3$.

A. $y = 64 - 48x + 12x^2 - x^3$

B. $y = 64 + 48x + 12x^2 + x^3$

C. $y = 192 + 144x + 36x^2 - 3x^3$

D. $y = 192 - 144x + 36x^2 - 3x^3$

5. A particle is initially at $x = -1$. Its motion has a velocity of $v = \dfrac{1}{t+e}$.

Which of the folling will be true when the particle is at the origin?

A. $t = e^{-1}$, $v = 1$

B. $t = 1$, $v = e^{-1}$

C. $t = e^{-2}$, $v = -1$

D. $t = e^2 - \text{e}$, $v = e^{-2}$

6. What is the domain and range of $y = 4cos^{-1}\dfrac{3x}{2}$?

A. Domain: $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$

Range: $-4\pi \leq y \leq 4\pi$.

B. Domain: $-\dfrac{3}{2} \leq x \leq \dfrac{3}{2}$

Range: $-4\pi \leq y \leq 4\pi$.

C. Domain: $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$

Range: $0 \leq y \leq 4\pi$.

D. Domain: $-\dfrac{2}{3} \leq x \leq \dfrac{2}{3}$

Range: $0 \leq y \leq 4$.

7. The angle between two vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is approximately.

A. $cos^{-1}(.5)$

B. $cos^{-1}(-.5)$

C. $cos^{-1}(1)$

D. $cos^{-1}(-1)$

8. X,Y, and Z are collinear points, with position vectors $\underset{\sim}{\mathbf{x}}$, $\underset{\sim}{\mathbf{y}}$ and $\underset{\sim}{\mathbf{z}}$ respectively.
Y lies between X and Z.
Given that $|\overrightarrow{YZ}| = \frac{1}{2}|\overrightarrow{XY}|$, which of the following expressions is equal to $\underset{\sim}{\mathbf{z}}$?

A. $\frac{1}{2}\underset{\sim}{\mathbf{x}} - \frac{3}{2}\underset{\sim}{\mathbf{y}}$

B. $\frac{3}{2}\underset{\sim}{\mathbf{x}} - \frac{1}{2}\underset{\sim}{\mathbf{y}}$

C. $\frac{3}{2}\underset{\sim}{\mathbf{y}} - \frac{3}{2}\underset{\sim}{\mathbf{x}}$

D. $\frac{3}{2}\underset{\sim}{\mathbf{y}} - \frac{1}{2}\underset{\sim}{\mathbf{x}}$

9. Which of the following is the derivatve of $y = x^2\, sin^{-1}\,(2x)$?

A. $2xsin^{-1}\,(2x) + \dfrac{2x^2}{\sqrt{1 - 4x^2}}$

B. $2xsin^{-1}\,(2x) + \dfrac{x^2}{\sqrt{1 - 4x^2}}$

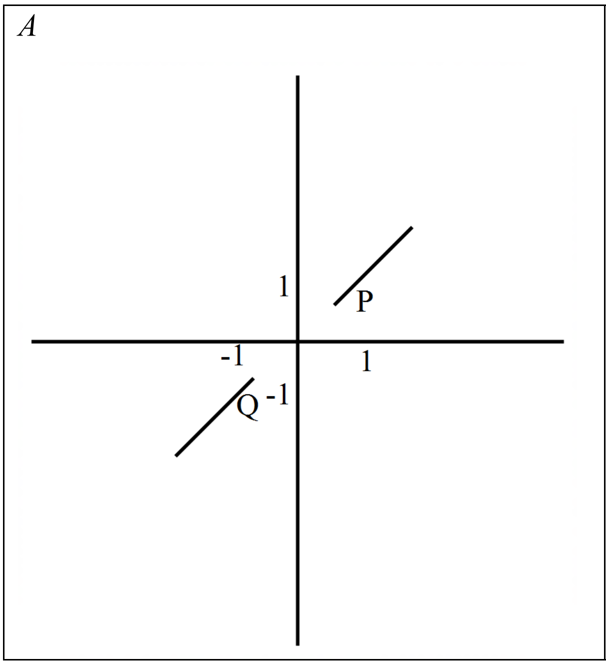
C. $2xsin^{-1}\,(2x) + \dfrac{2x^2}{\sqrt{1 - 2x^2}}$

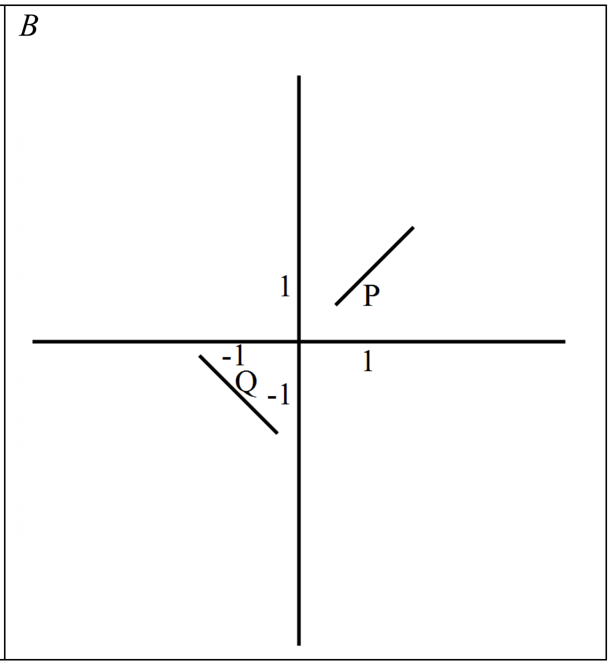
D. $2xsin^{-1}\,(2x) + \dfrac{x^2}{\sqrt{1 - 2x^2}}$

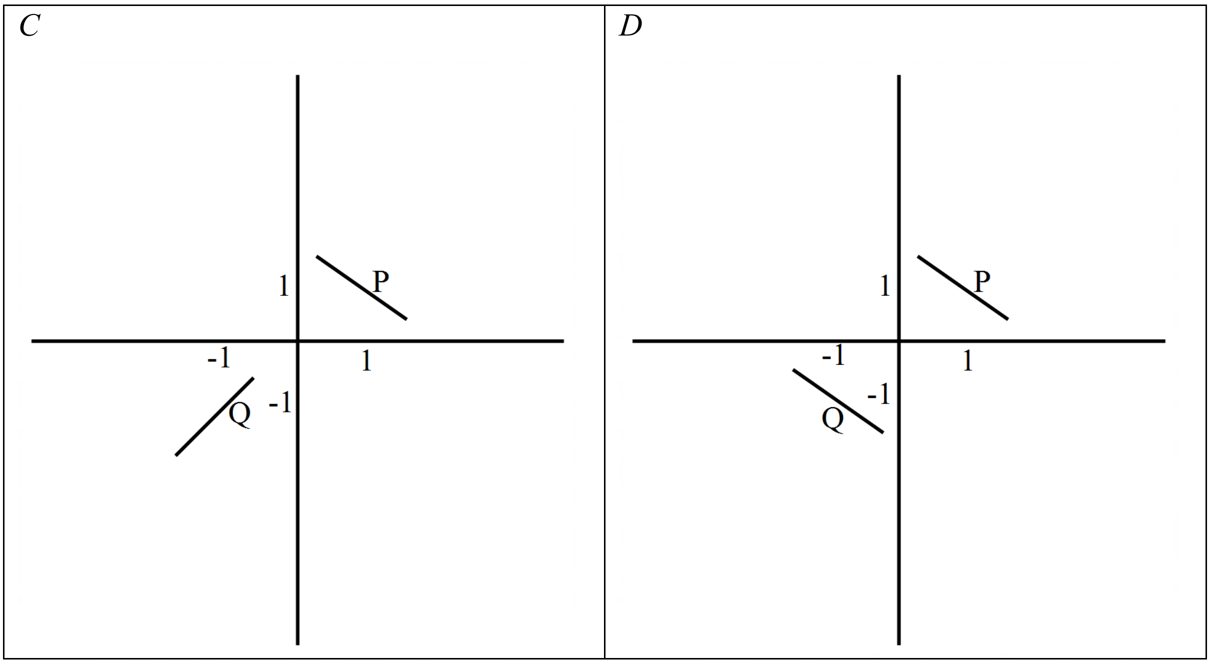
10. A directional field is to be drawn for the differential equation,

$$\frac{dy}{dx} = 2x + \frac{y}{x}.$$

Which of the following graphs shows the correct slope lines at the points $P(1,1)$ and $Q(-1,-1)$?







End of Section 1

Section II **60 marks** **Attempt Questions 11-14**

Answer each question in the appropriate writing booklet.

Extra exam writing booklets are available.

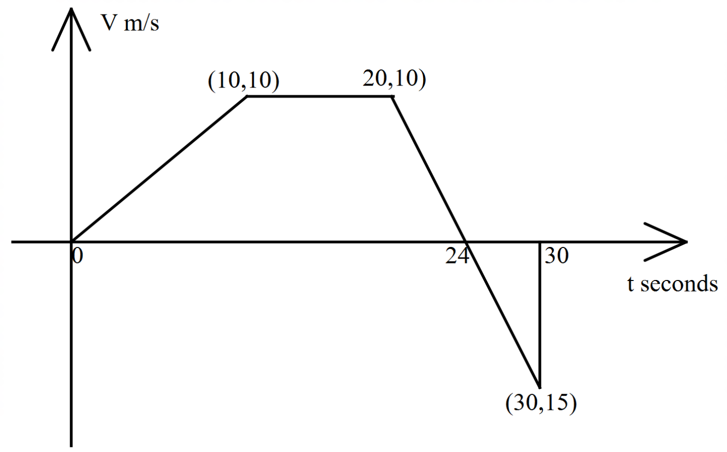
Question 11 **Start on a new answer booklet** **15 Marks.**

(a) Solve $|1 - 2x| \geq 15$. 2

(b) Given the velocity time graph below find:

(i) The acceleration for the first 10 seconds 1

(ii) The total distance travelled in the first 30 seconds. 1



(c) Use the substitution $u = e^{2x}$ to find $\int \frac{e^{2x}}{9 + e^{4x}}\, dx$. 3

(d) Water is poured into a container at a rate of $8\ cm^3/s$. If the volume of the water in the container is given by $V = \frac{3}{2}(h^2 + 8h)\ cm^3$ where h is the depth of the water, find the rate of change of the depth of the water when $V = 72\ cm^3$. 4

(e) (i) Express $cosx - \sqrt{3}\, sinx$ in the form $Rcos(x + \alpha)$ where $R > 0$ and $0 \leq x \leq 2\pi$. 2

(ii) Find the co-ordinates of the points of intersection of the graph $y = cosx - \sqrt{3}\, sinx$ with the x and y axis, in the interval $0 \leq x \leq 2\pi$. 2

End of Question 11

Question 12 **Start on a new answer booklet** **15 Marks.**

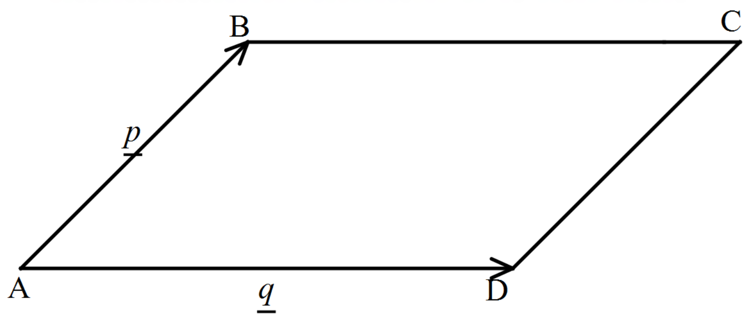
(a) Prove that $\frac{sin\ 3x}{sinx} - \frac{cos\ 3x}{cosx} = 2$. 3

(b) Use the substitution $x = 2sin\theta$ to find the exact value of $\int_0^1 \sqrt{4 - x^2}\, dx$. 3

(c) The vectors $\underset{\sim}{\mathbf{u}} = \begin{pmatrix} 3 \\ sin\ 2\alpha \end{pmatrix}$ and $\underset{\sim}{\mathbf{v}} = \begin{pmatrix} cos\alpha \\ -3 \end{pmatrix}$, where $0 < \alpha < \pi$, are perpendicular. Find possible values of α. 3

(d) Let $P(x) = x^5 - 6x^3 - 8x^2 - 3x$.
Show that $x = -1$ is a root of $P(x)$ of multiplicity three. 3

(e) $ABCD$ is a parallelogram, where $\overrightarrow{AB} = \underset{\sim}{\mathbf{p}}$ and $\overrightarrow{AD} = \underset{\sim}{\mathbf{q}}$.



Prove, using vectors, that the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of all the four sides. 3

End of Question 12

Question 13 Start on a new answer booklet 15 Marks.

(a) (i) Show that $sin(x - y)sin(x + y) = sin^2x - sin^2y$. 1

(ii) Hence or otherwise,
Solve $sin^2 3x - sin^2x = sin\, 4x$. $x = [0\, ,\, \pi]$ 2

(b) (i) Sketch the graph of $y = f(x) = x^2 - 1$. 1

(ii) On the same graph sketch $\dfrac{1}{x^2 - 1} + 2$.
Clearly identifying asymptotes and intercepts. 3

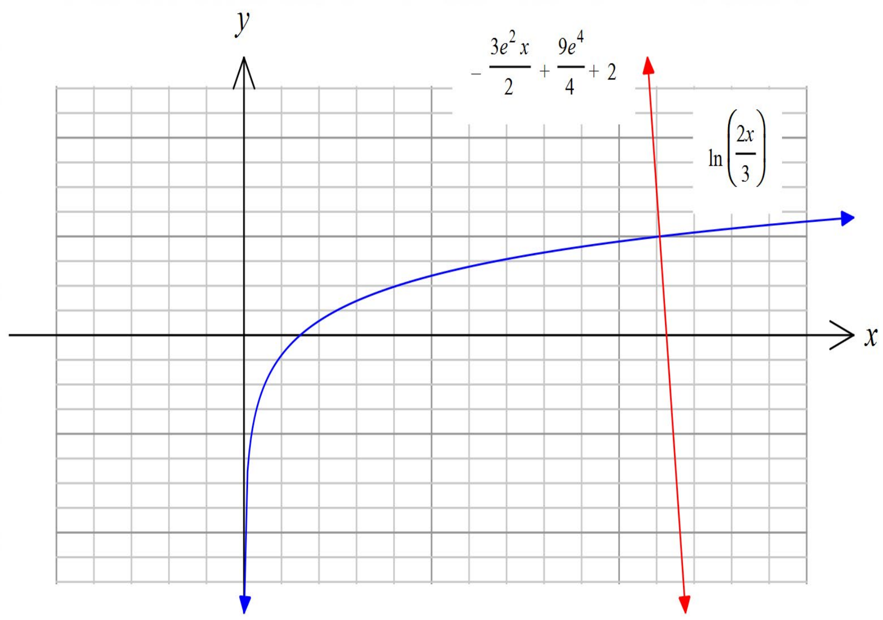
(c) Given $|\underset{\sim}{\mathbf{p}}| = 3$, $|\underset{\sim}{\mathbf{q}}| = 5$ and $\underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{q}} = 6$.
Calculate the length of $2\underset{\sim}{\mathbf{p}} - 3\underset{\sim}{\mathbf{q}}$. 3

(d) Let $f(x) = ln\left(\frac{2x}{3}\right)$.

(i) Show that the equation of the normal to the graph of $y = f(x)$ at the point where $x = \frac{3e^2}{2}$ has the equation $y = -\frac{3e^2x}{2} + \frac{9e^4}{4} + 2$. 2

(ii) Find the exact area enclosed by the graph of $y = f(x)$, the line $y = -\frac{3e^2x}{2} + \frac{9e^4}{4} + 2$ and the x axis. 3

(You may use the diagram below to help solve the problem)



End of Question 13

Question 14 **Start on a new answer booklet** **15 Marks.**

(a) (i) Show that $\frac{d}{dx}(tan^3x) = 3sec^4x - 3sec^2x$. 1

(ii) Hence or otherwise, evaluate $\int_0^{\frac{\pi}{4}} sec^4x dx$. 3

(b) Prove by mathematical induction for all integers $n \geq 1$ that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$. 3

(c) Given that $\dfrac{dy}{dx} = \dfrac{1}{4}(y - 1)^2$ and $y(0) = 0$.
What is the value of x when $y = 2$? 3

(d) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2}{x^3 e^y}$ given $y(1) = 0$.
Express your solution in the form $y = f(x)$. 3

(e) Let $\underset{\sim}{\mathbf{g}} = |\underset{\sim}{\mathbf{e}}|\underset{\sim}{\mathbf{f}} + |\underset{\sim}{\mathbf{f}}|\underset{\sim}{\mathbf{e}}$ where $\underset{\sim}{\mathbf{e}}$, $\underset{\sim}{\mathbf{f}}$ and $\underset{\sim}{\mathbf{g}}$ are non-zero vectors.
Show that $\underset{\sim}{\mathbf{g}}$ bisects the angle between $\underset{\sim}{\mathbf{e}}$ and $\underset{\sim}{\mathbf{f}}$. 2

End of Question 14

END OF TASK

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2024 Trial HSC Examination
Mathematics Extension 1 Course

Name Solutions' **Teacher** **REM** **DPR** **NHM** **JPN**

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ● C ○ D ○

If you change your mind and have crossed out what you consider the correct answer, indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ● B ● C ○ D ○

1.	A ●	B ○	C ○	D ○
2.	A ●	B ○	C ○	D ○
3.	A ○	B ○	C ●	D ○
4.	A ○	B ○	C ○	D ●
5.	A ○	B ○	C ○	D ●
6.	A ○	B ○	C ●	D ○
7.	A ●	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ●
9.	A ●	B ○	C ○	D ○
10.	A ○	B ●	C ○	D ○

Start your answer here. Question 11

(a) Solve $|1-2x| \geq 15$

$1-2x = -15$ $\qquad$ $1-2x = 15$

$-2x = -16$ $\qquad$ $-2x = 14$

$x = 8$ $\qquad$ $x = -7$

$\therefore \boxed{x \leq -7} \checkmark \cup \boxed{x \geq 8} \checkmark$

(b) (i) $a = \frac{v}{t} = \frac{10}{10} = \boxed{1\ m/s^2}$ $\checkmark$

(ii) Distance travelled is Area A+B

A = trapezium $= \frac{1}{2}(10+24)\times 10$

$= \frac{1}{2}\, 34 \times 10 = 170$

B: Triangle $= \frac{1}{2} \times 6 \times 15 = 45$

$\therefore$ Distance travelled is $170+45$

$= \boxed{215\text{ m.}}$ $\checkmark$

Many did not understand how to manipulate the inequalities

ie $1-2x \geq 15$ or $-1+2x \geq 15$

1 mark for each correct solution

1 mark

Distance not displacement

$\therefore$ total area above and below axis

1 mark

(c) $u = e^{2x}$ $\quad du = 2e^{2x}\,dx$

$\frac{1}{2}du = e^{2x}\,dx$ ✓ correct organisation

$\therefore \int \frac{e^{2x}\,dx}{9+e^{2x}} = \frac{1}{2}\int \frac{du}{9+u^2}$ ✓ correct integral in terms of u

$= \frac{1}{2}\int \frac{du}{3^2+u^2} = \frac{1}{6}\int \frac{3\,du}{3^2+u^2}$

some did not realise that it involved inverse tan

$= \frac{1}{6}\tan^{-1}\left(\frac{u}{3}\right) + C$

$= \frac{1}{6}\tan^{-1}\left(\frac{e^{2x}}{3}\right) + C.$ ✓ correct answer

(d) Given $\frac{dV}{dt} = 8\text{ cm}^3/\text{s}$ $\quad V = \frac{3}{2}(h^2+8h)\text{ cm}^3$

Find $\frac{dh}{dt}$, when $V = 72\text{ cm}^3$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\qquad \frac{dV}{dh} = \frac{3}{2}(2h+8)$

$= 3h+12$ ✓ Finding correct expression for $\frac{dV}{dh}$

$\therefore 8 = (3h+12) \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{8}{3(h+4)}$ ✓ correct Expression for $\frac{dh}{dt}$

To find 'h' $\quad V = \frac{3}{2}(h^2+8h) = 72$

$h^2+8h = 48 \qquad h^2+8h-48 = 0$ finding value of h

$(h-4)(h+12) = 0 \qquad h = 4,\ h = -12 \quad \therefore h = 4$ ✓

$\therefore \frac{dh}{dt} = \frac{8}{3(4+8)} = \boxed{\frac{1}{3}\text{ cm/s.}}$ ✓ calculating value of $\frac{dh}{dt}$

Additional writing space on back page.

(e)(i) Express $\cos x - \sqrt{3}\sin x \equiv R\cos(x+\alpha)$

$1\cos x - \sqrt{3}\sin x \qquad R = \sqrt{1^2+(\sqrt{3})^2} = 2$ — 1 finding r

$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$ (acute angle only)

$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x+\frac{\pi}{3})$ — 1 finding α

(ii)

For points of intersection:

at $x=0 \quad y = 2\cos(\frac{\pi}{3}) = 2\times\frac{1}{2} = 1$ — $\boxed{(0,1)}$

finding y intercept

at $y=0 \quad 2\cos(x+\frac{\pi}{3}) = 0$

$\cos(x+\frac{\pi}{3}) = 0 \qquad 0 \le x \le 2\pi$

$x+\frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$x = \frac{\pi}{2}-\frac{\pi}{3}, \frac{3\pi}{2}-\frac{\pi}{3}, \frac{5\pi}{2}-\frac{\pi}{3}, \frac{7\pi}{2}-\frac{\pi}{3}, \ldots$

$x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6} \ldots$ out of domain

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}$

$\therefore \boxed{(\frac{\pi}{6},0), (\frac{7\pi}{6},0)}$

finding co-ords of x intercepts

You may ask for an extra Writing Booklet if you need more space to answer this question.

Start your answer here. Question 12

(a) $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$

$= \frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x} = \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x}$

$= \frac{2 \sin 2x}{\sin 2x} = 2$

$\sin x \cos x$

$= \frac{1}{2} \times 2 \sin x \cos x$

$= \frac{1}{2} \sin 2x$

* Well Answered

* Most students split $\sin 3x$ and $\cos 3x$ into $\sin(2x+x)$ and $\cos(2x+x)$ and use appropriate Trig identities

(b) $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$ when $x = 0$, $2\sin\theta = 0$, $\theta = 0$

$x = 1$, $2\sin\theta = 1$ $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$\therefore \int_0^1 \sqrt{4 - x^2}\, dx = \int_0^{\pi/6} \sqrt{4 - 4\sin^2\theta} \times 2\cos\theta \, d\theta$

$= \int_0^{\pi/6} 2\cos\theta \times 2\cos\theta \, d\theta$

$= 4\int_0^{\pi/6} \cos^2\theta \, d\theta = 4 \times \frac{1}{2}\int_0^{\pi/6} 1 + \cos 2\theta \, d\theta$

$\therefore 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6} = 2\left[\frac{\pi}{6} + \frac{\sqrt{3}}{2 \times 2} - 0 - 0\right]$

$= 2\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right] = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

* Poorly answered.

* Many students did not express dx in terms of $d\theta$

* Many students did not change the limits.

(c) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ for perpendicular

$$\begin{pmatrix} 3 \\ \sin 2\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha \\ -3 \end{pmatrix} = 0$$

$3\cos\alpha - 3\sin 2\alpha = 0$

$3\cos\alpha - 6\sin\alpha\cos\alpha = 0$

$3\cos\alpha(1 - 2\sin\alpha) = 0$

$\cos\alpha = 0 \quad , \quad 1 - 2\sin\alpha = 0$

$\cos\alpha = 0 \qquad \sin\alpha = \frac{1}{2}$

$$\left.\begin{array}{l} \text{For } \cos\alpha = 0 \quad \alpha = 0, \pi/2 \\ \text{For } \sin\alpha = \frac{1}{2} \quad \alpha = \frac{\pi}{6}, \frac{5\pi}{6} \end{array}\right\} \; 0 < \alpha < \pi$$

$\therefore \alpha = \pi/6, \pi/2, 5\pi/6$

* Well answered.

* Some students included $\frac{3\pi}{2}$ in their solutions and forgot the limit for α was $0 < \alpha < \pi$

* Many students forgot to include $\frac{5\pi}{6}$ in the solution set.

(d) $P(x) = x^5 - 6x^3 - 8x^2 - 3x$

$P'(x) = 5x^4 - 18x^2 - 16x - 3$

$P''(x) = 20x^3 - 36x - 16$

To prove multiplicity of roots.

$P(x) = P'(x) = P''(x)$ at $x = -1$ (Given).

$\therefore P(-1) = -1 + 6 - 8 + 3 = 0$

$P'(-1) = 5 - 18 + 16 - 3 = 0$

$P''(-1) = -20 + 36 - 16 = 0$

$\therefore x = -1$ is a root of multiplicity three

or $(x+1)^3$ is a factor of $P(x)$.

* Reasonably well answered.

* Some students divided $P(x)$ by $x+1$ and the divided the dividend by $x+1$ and repeated the process until such time as they were able to express in the form $(x+1)^3 Q(x)$

If you did this correctly you get full marks.

Additional writing space on back page.

(e) $\overline{AD} = \overline{BC} = \underset{\sim}{q}$

$\overline{AB} = \overline{DC} = \underset{\sim}{p}$

$\overline{AC} = \underset{\sim}{p} + \underset{\sim}{q}$, $\overline{BD} = \underset{\sim}{p} - \underset{\sim}{q}$

$\therefore$ Sum of squares of diagonal

$|\underset{\sim}{p}+\underset{\sim}{q}|^2 + |\underset{\sim}{p}-\underset{\sim}{q}|^2$

$= (\underset{\sim}{p}+\underset{\sim}{q}) \cdot (\underset{\sim}{p}+\underset{\sim}{q}) + (\underset{\sim}{p}-\underset{\sim}{q})(\underset{\sim}{p}-\underset{\sim}{q})$

$= \underset{\sim}{p}^2 + 2\underset{\sim}{p}\underset{\sim}{q} + \underset{\sim}{q}^2 + \underset{\sim}{p}^2 - 2\underset{\sim}{p}\underset{\sim}{q} + \underset{\sim}{q}^2$

$= \underset{\sim}{p}^2 + \underset{\sim}{q}^2 + \underset{\sim}{p}^2 + \underset{\sim}{q}^2$

= Sum of the squares of all four sides.

* Well answered.

You may ask for an extra Writing Booklet if you need more space to answer this question.

Start your answer here. Question 13

(a) (i) $\sin(x-y)\sin(x+y)$

$= \frac{1}{2}\left[\cos[(x-y)-(x+y)] - \cos[(x-y)+(x+y)]\right]$

$= \frac{1}{2}\left[\cos(-2y) - \cos(2x)\right]$

$= \frac{1}{2}\left(\cos 2y - \cos 2x\right)$

$= \frac{1}{2}\left[1-2\sin^2 y - (1-2\sin^2 x)\right]$ ✓

$= \frac{1}{2}\left[-2\sin^2 y + 2\sin^2 x\right]$

$= \sin^2 x - \sin^2 y.$

(ii) From (i) $\sin^2 3x - \sin^2 x$

$= \sin(3x-x)\sin(3x+x)$

$= \sin 2x \sin 4x$

$\therefore \quad \sin^2 3x - \sin^2 x = \sin 4x$

ie $\quad \sin 2x \sin 4x = \sin 4x$ from (i)

$\sin 2x \sin 4x - \sin 4x = 0$

$\sin 4x\left[\sin 2x - 1\right] = 0$ ✓

ie $\sin 4x = 0 \qquad \sin 2x = 1$

For $\sin 4x = 0 \qquad 4x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \qquad 0 \le x \le \pi.$

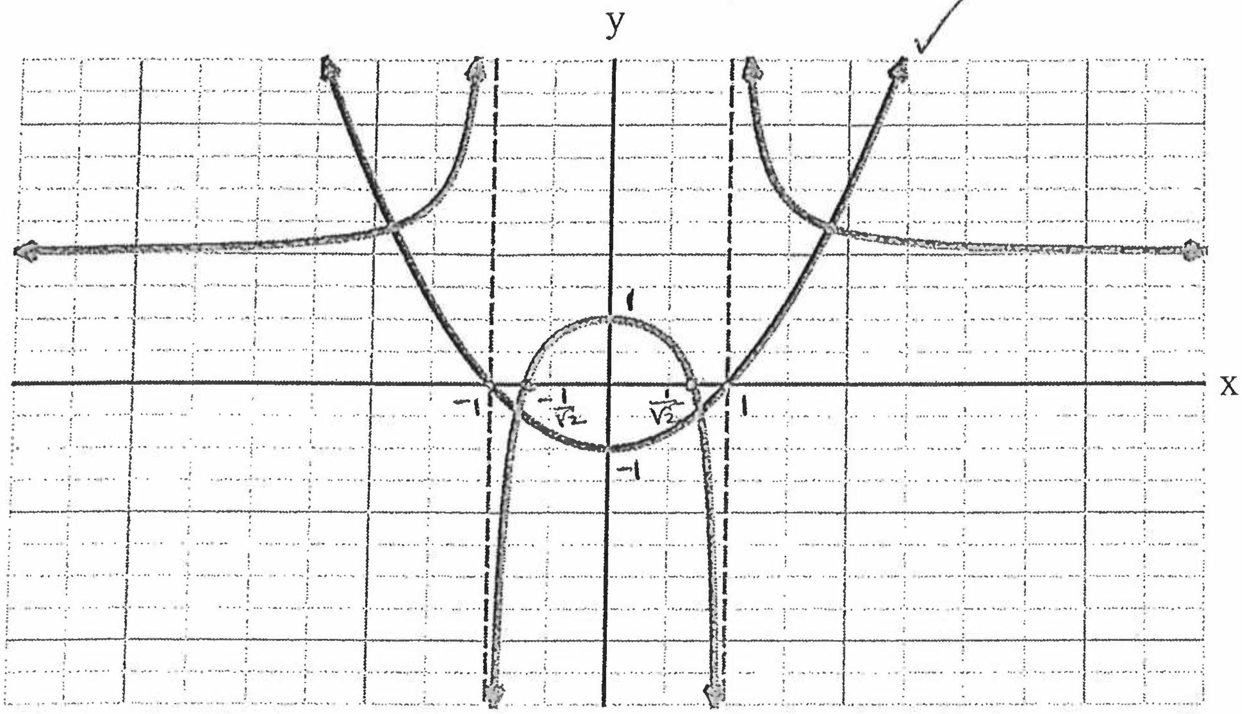
$\sin 2x = 1, \quad 2x = \pi/2, \quad x = \frac{\pi}{4}$ ✓

$\therefore$ Solution: $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi.$ ✓

Solutions to 13 (b) (i) and (ii)

(*i*) Sketch the graph of $y = f(x) = x^2 - 1$.

(*ii*) On the same graph sketch $\dfrac{1}{x^2 - 1} + 2$.



for x-intercepts.

$$\frac{1}{x^2-1} + 2 = 0$$

$$\frac{1}{x^2-1} = -2$$

$$1 = -2(x^2-1)$$

$$1 = -2x^2 + 2$$

$$\therefore 2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

NOTE:

Most students lost a mark for not getting the x intercept

1 Mark Correct Shape

1 " Asymptotes

$x = \pm 1$

$y = 2$

1 Mark correct intercepts

(c) $|p| = 3$, $|q| = 5$ $p \cdot q = 6$

Formula $|a - b| = \sqrt{(a-b)(a-b)}$

$\therefore |2p - 3q| = \sqrt{(2p-3q)\cdot(2p-3q)}$

$= \sqrt{2p \cdot 2p - 2p \cdot 3q - 3q \cdot 2p + 3q \cdot 3q}$

$= \sqrt{4(p \cdot p) - 6(p \cdot q) - 6(q \cdot p) + 9(q \cdot q)}$

$p \cdot p = |p|^2 = 3^2 = 9$

$q \cdot q = |q|^2 = 5^2 = 25$

$p \cdot q = 6$

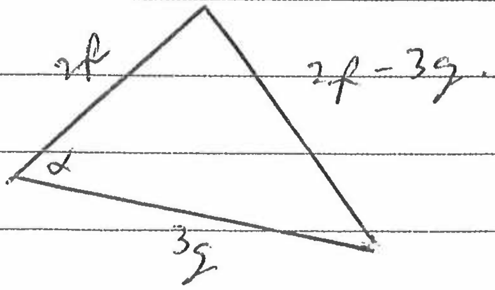
$= \sqrt{4(9) - 12(6) + 9(25)}$

$= \sqrt{189}$

$\therefore |2p - 3q| = \sqrt{189} = \sqrt{9 \times 21}$

$= 3\sqrt{21}$

OR



$|2p - 3q|^2 = |2p|^2 + |3q|^2 - 2|2p||3q| \cos\alpha$

$= 4|p|^2 + 9|q|^2 - 2 \times 2|p| \times 3 \times |q| \dfrac{2p \cdot 3q}{|2p||3q|}$

$= 4(3)^2 + 9(5)^2 - 2 \times 2 \times (3) \times 3 \times (5) \left(\dfrac{6 \times 6}{2 \times 3 \times 3 \times 5} \right)$

Additional writing space on back page.

$= 189$

length $2p - 3q = \sqrt{189}$

$= 3\sqrt{21}$

(d) $f(x) = \ln\left(\frac{2x}{3}\right)$

(i) $y = \ln 2x - \ln 3$

$y' = \frac{1}{2x} \times 2 = \frac{1}{x}$

$y = \ln\left(\frac{2x}{3}\right)$

@ $x = \frac{3e^2}{2}$

$y = \ln\left(\frac{2}{3} \times \frac{3e^2}{2}\right)$

$= \ln e^2$

$= 2\ln e = 2$

$m_t = \frac{1}{x} = \frac{2}{3e^2}$

$\therefore m_n = -\frac{3e^2}{2}$

1 Mark

Equn. of normal at $\left(\frac{3e^2}{2}, 2\right)$ is:

$y - y_1 = m_n(x - x_1)$

$y - 2 = -\frac{3e^2}{2}\left(x - \frac{3e^2}{2}\right)$

1 Mark

$y = -\frac{3e^2}{2}x + \frac{9e^4}{4} + 2$

(ii) Area of shaded region

See diagram.

$= A(\text{Trapezium OABC}) - \int_0^2 \frac{3}{2}e^y\,dy$

$y = -\frac{3e^2}{2}x + \frac{9e^4}{4} + 2$

$0 = -\frac{3e^2}{2}x + \frac{9e^4}{4} + 2$

$\frac{3e^2}{2}x = \frac{9e^4}{4} + 2$

$x = \frac{2}{3e^2} \times \frac{9e^4}{4} + \frac{2 \times 2}{3e^2}$

$x = \frac{3e^2}{2} + \frac{4}{3e^2}$

$A(\text{Trap}) = \frac{1}{2} \times \left(\frac{3e^2}{2} + \frac{3e^2}{2} + \frac{4}{3e^2}\right) \times 2$

$= 3e^2 + \frac{4}{3e^2}$

Additional writing space on back page.

$y = \ln\left(\frac{3x}{3}\right)$

$e^y = \frac{3x}{3}$

$\frac{3}{3} e^y = x$

$A_{\text{curve}} = \int_0^2 \frac{3}{2} e^y \, dy$

$= \left[\frac{3}{2} e^y\right]_0^2 = \frac{3}{2}e^2 - \frac{3}{2}$

c. Area Shaded $= 3e^2 + \frac{4}{3e^2} - \left(\frac{3}{2}e^2 - \frac{3}{2}\right)$

$= 3e^2 + \frac{4}{3e^2} - \frac{3e^2}{2} + \frac{3}{2}$

$= \frac{3e^2}{2} + \frac{4}{3e^2} + \frac{3}{2}\ u^2.$

You may ask for an extra Writing Booklet if you need more space to answer this question.

$y = \ln\left(\frac{2x}{3}\right)$

$y' = \frac{1 \times 3 \times 2}{2x \times 3} = \frac{1}{x}$

$m_t = \frac{1}{x} = \frac{2}{3e^2}$

$m_n = -\frac{3e^2}{2}$

$y = \ln\left(\frac{2x}{3}\right)$

$y = \ln\left(\frac{2}{3} \times \frac{3}{2} e^2\right)$

$y = \ln e^2$

$y = 2$

$y - 2 = -\frac{3e^2}{2}\left(x - \frac{3e^2}{2}\right)$

$y = -\frac{3e^2 x}{2} + \frac{9e^4}{4} + 2$

✓✓

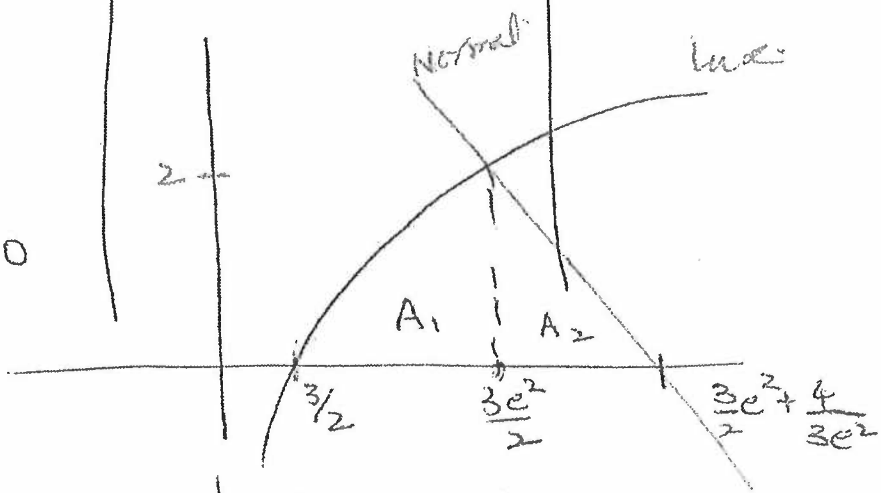
Area

at $y = 0$ Curve.

$\ln\left(\frac{2x}{3}\right) = 0$

$e^0 = \frac{2x}{3}$

$x = \frac{3}{2}$



at $y = 0$ Normal.

$\frac{3e^2 x}{2} = \frac{9e^4}{4} + 2$

$x = \frac{\cancel{9}^{3} \times 2}{\cancel{3} \times \cancel{4}_{2}} \times \frac{e^4}{e^2} + \frac{2 \times 2}{3e^2}$

$x = \frac{3e^2}{2} + \frac{4}{3e^2}$

at $y = 2$ Curve.

$2 = \ln\left(\frac{2x}{3}\right)$

$e^2 = \frac{2x}{3}$

$x = \frac{3e^2}{2}$

Area = A_1 $\int_{3/2}^{\frac{3e^2}{2}} \ln\left(\frac{2x}{3}\right) dx + \frac{1}{2} \times b \times h.$

$+ \frac{1}{\cancel{2}}\left(\frac{3e^2}{2} + \frac{4}{3e^2} - \frac{3e^2}{2}\right) \times \cancel{2}$

$+ \frac{4}{3e^2}$

$\downarrow$

Int. by Parts

Next Page.

$$A_1: \int_{3/2}^{3e^2/2} \ln\left(\frac{2x}{3}\right) dx$$

$$= \int \ln 2x - \ln 3 \, dx$$

$$= \int (\ln x + \ln 2 - \ln 3) dx$$

$$\therefore \left[x\ln x - x + x\ln 2 - x\ln 3\right]_{3/2}^{3e^2/2}$$

$$\frac{3e^2}{2}\ln\left(\frac{3e^2}{2}\right) - \frac{3e^2}{2} + \frac{3e^2}{2}\ln 2 - \frac{3e^2}{2}\ln 3$$

$$- \frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{3}{2} - \frac{3}{2}\ln 2 + \frac{3}{2}\ln 3$$

$$= \frac{3e^2}{2}(\ln 3 + 2 - \ln 2) - \frac{3e^2}{2} + \frac{3e^2}{2}\ln 2 - \frac{3e^2}{2}\ln 3$$

$$- \frac{3}{2}(\ln 3 - \ln 2) + \frac{3}{2} - \frac{3}{2}\ln 2 + \frac{3}{2}\ln 3$$

$$= \cancel{\frac{3e^2}{2}\ln 3} + \frac{3e^2}{2} \cdot 2 - \cancel{\frac{3e^2}{2}\ln 2} - \frac{3e^2}{2} + \cancel{\frac{3e^2}{2}\ln 2} - \cancel{\frac{3e^2}{2}\ln 3}$$

$$- \cancel{\frac{3}{2}\ln 3} + \cancel{\frac{3}{2}\ln 2} + \frac{3}{2} - \cancel{\frac{3}{2}\ln 2} + \cancel{\frac{3}{2}\ln 3}$$

$$= \frac{3e^2}{2} + \frac{3}{2}$$

$$\therefore = A_1 + A_2$$

$$= \frac{3e^2}{2} + \frac{3}{2} + \frac{4}{3e^2}$$

$$\int \ln x \, dx = x\ln x - x$$

$$\int \ln 2 = x\ln 2$$

$$\int \ln 3 = x\ln 3$$

$$\ln\left(\frac{3e^2}{2}\right)$$

$$= \ln 3e^2 - \ln 2$$

$$= \ln 3 + \ln e^2 - \ln 2$$

$$= \ln 3 + 2 - \ln 2$$

QUESTION 14

a) i) $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x(\sec^2 x)$

$= 3(\sec^2 x - 1)(\sec^2 x)$

$= 3\sec^4 x - 3\sec^2 x$

Surprisingly, many mocked this up.

1mk

(ii) $\int_0^{\frac{\pi}{4}} \sec^2\theta = ??$

From (i)

$3\sec^4 x - 3\sec^2 x = \frac{d}{dx}(\tan^3 x)$

$3\sec^4 x = \frac{d}{dx}(\tan^3 x) + 3\sec^2 x \leftarrow$ 1mk

$\sec^4 x = \frac{1}{3}\left[\frac{d}{dx}(\tan^3 x)\right] + \sec^2 x$

$\therefore \int_0^{\frac{\pi}{4}} \sec^4 x\,dx = \left[\frac{1}{3}\tan^3 x + \tan x\right]_0^{\frac{\pi}{4}} \leftarrow$ 1mk

$= \left(\frac{1}{3}\tan^3\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - (0)$

$= \frac{1}{3} + 1$

$= \frac{4}{3} \leftarrow$ 1mk

Poorly done

Many students "lost" the 3.

b) Prove $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\dots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$

for $n \geq 1$

Step 1. Prove true for $n=1$.

LHS $=\frac{1}{2}$

RHS $=2-\frac{3}{2}$

$=\frac{1}{2}$

$\therefore$ true for $n=1$

1mk for correct setting out

Step 2. Assume true for $n=k$.

ie $\frac{1}{2}+\frac{2}{2^2}+\dots+\frac{k}{2^k}=2-\frac{k+2}{2^k}$

Now Prove true for $n=k+1$

ie $\frac{1}{2}+\frac{2}{2^2}+\dots+\frac{k+1}{2^{k+1}}=2-\frac{k+3}{2^{k+1}}$

1mk for correct setting out

Proof.

LHS $=\frac{1}{2}+\dots+\frac{k}{2^k}+\frac{k+1}{2^{k+1}}$

$=2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$ From assumption

$=\frac{2(2^{k+1})-2(k+2)+k+1}{2^{k+1}}$

$=\frac{2(2^{k+1})}{2^{k+1}}-\frac{k+3}{2^{k+1}}$

$=2-\frac{k+3}{2^{k+1}}$ etc by MI

1mk for use of assumption & correct working to soln.

c) $\frac{dy}{dx} = \frac{1}{4}(y-1)^2$

METHOD 1

$\frac{dy}{(y-1)^2} = \frac{1}{4}\,dx$

$(y-1)^{-2}\,dy = \frac{1}{4}\,dx$

$\frac{(y-1)^{-1}}{-1} = \frac{x}{4} + c$

$-\frac{1}{y-1} = \frac{x}{4} + c$

$x = 0 \quad y = 0$

$1 = c$

$-\frac{1}{y-1} = \frac{x}{4} + 1$

$y = 2 \quad x = ???$

$-\frac{1}{1} = \frac{x}{4} + 1$

$x = -8$

METHOD 2

$\frac{dx}{dy} = 4(y-1)^{-2}$

$x = -4(y-1)^{-1} + c$

$x = \frac{-4}{y-1} + c$

$x = 0 \quad y = 0$

$0 = -\frac{4}{-1} + c$

$c = -4$

$x = -\frac{4}{y-1} - 4$

$y = 2 \quad x = ??$

$x = -\frac{4}{1} - 4$

$= -8$

1mk for either correct method

1mk for correct constant. (many forgot to allow for constant)

1mk for correct solution

d) $\dfrac{dy}{dx} = \dfrac{2}{x^3 e^y}$ $x = 1$ $y = 0$

$$e^y \, dy = \frac{2}{x^3} \, dx$$

1mk for correct method.

$$e^y \, dy = 2x^{-3} \, dx$$

$$e^y = \frac{2x^{-2}}{-2} + c$$

$$e^y = -\frac{1}{x^2} + c$$

Again, many forgot the "c"

$x = 1$ $y = 0$

$$e^0 = -1 + c$$

$$c = 2$$

1mk for value of "c".

$$\therefore e^y = -\frac{1}{x^2} + 2$$

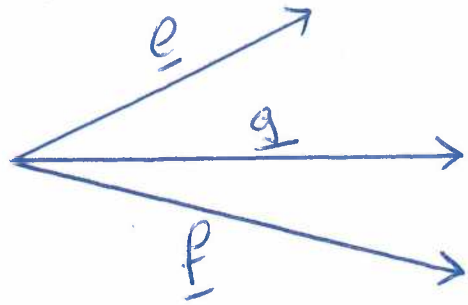
$$y = \ln\left(2 - \frac{1}{x^2}\right)$$

1mk for correct solution

e) One path to a correct solution is using the dot product

$\underline{u} . \underline{v} = |\underline{u}||\underline{v}| \cos\theta$

$\underline{g} = |\underline{e}|\underline{f} + |\underline{f}|\underline{e}$



1mk awarded for substantial effort

$$\underline{e} . \underline{g} = \underline{e} . \left[|\underline{e}|\underline{f} + |\underline{f}|\underline{e} \right]$$

$$= |\underline{e}|\, \underline{e}.\underline{f} + |\underline{f}|\, \underline{e}.\underline{e}$$

$$= |\underline{e}|\, \underline{e}\,\underline{f} + |\underline{f}|\, |\underline{e}|^2$$

$$= |\underline{e}| \left[\underline{e}\,\underline{f} + |\underline{f}|\,|\underline{e}| \right]$$

$$\therefore |\underline{e}| . |\underline{g}| \cos\alpha = |\underline{e}| \left[\underline{e}\,\underline{f} + |\underline{f}||\underline{e}| \right]$$

OR $|\underline{g}| \cos\alpha = \underline{e}.\underline{f} + |\underline{f}||\underline{e}|$ *

Similarly, $\underline{f} . \underline{g} = \underline{f} \left[|\underline{e}|\underline{f} + |\underline{f}|\underline{e} \right]$

$$= |\underline{e}||\underline{f}|^2 + |\underline{f}|\, \underline{e}\,\underline{f}$$

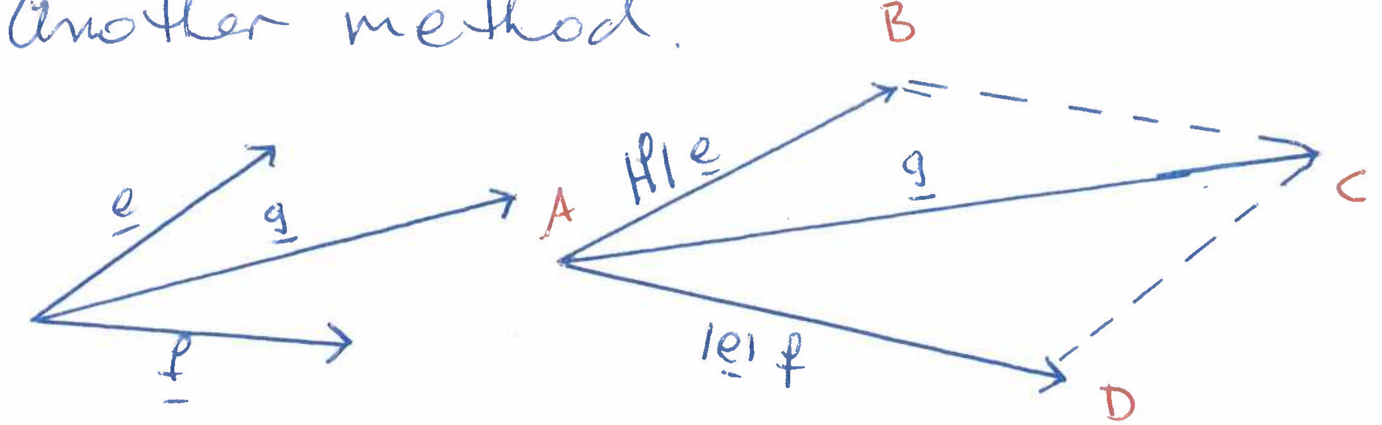
$$= |\underline{f}| \left[|\underline{e}||\underline{f}| + \underline{e}.\underline{f} \right]$$

$$\therefore |\underline{f}||\underline{g}| \cos\beta = |\underline{f}| \left[|\underline{e}||\underline{f}| + \underline{e}\,\underline{f} \right]$$

$$|\underline{g}| \cos\beta = |\underline{e}||\underline{f}| + \underline{e}\,\underline{f}$$ *

$\therefore \cos\alpha = \cos\beta$

e) Another method.



$\underline{g} = |\underline{f}|\,\underline{e} + |\underline{e}|\,\underline{f}$

Now $\left|\,|\underline{f}|\,\underline{e}\,\right| = \left|\,|\underline{e}|\,\underline{f}\,\right|$ * MUST BE REFERENCED IN SOLUTION

$\therefore$ ABCD is a Rhombus

If ABCD is a Rhombus

then $\underline{g}$ bisects the angle between

$|\underline{f}|\,\underline{e}$ and $|\underline{e}|\,\underline{f}$.

Hence, it bisects the angle between

$\underline{e}$ and $\underline{f}$

1 mk awarded for substantial effort